

Bulk arrival, fixed batch service queue with unreliable server and with Bernoulli vacation, Two stages of service

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ABSTRACT

A Poisson queue with batch arrival and two stages of batch service has been considered. In addition the server take Bernoulli vacation and the server breaks down and the services are given in two stages. For this model, using supplementary variable technique, the probability generating function of numbers of customers in the queue at various server states have been obtained. Some operating characteristics have been derived and numerical examples are given.

KEY WORDS: queuing; vacation; unreliable server; probability generating function and operating characteristics. AMS Subject classification number- 90B22, 60K25 and 60K30.

1. INTRODUCTION

The congestion situations encountered in computer, communication, manufacturing, production system, etc, can be modelled as queueing system with vacation. Several researchers have contributed significantly on vacation models [Takagi(1991), Lee, (1995), Bacot (2001) and Choudhury (2002), Ke (2007)]. Doshi (1986) and Takagi (1991) are the two excellent survey works on vacation queue. In many real life situations, the server may break down, so that a more realistic queueing model is that which incorporates the assumption of unreliable server. Many researchers have contributed on queue with unreliable servers [Li (1997), Wang((1995), Wang(1997)]. Some notable works on queueing with break down are Wang (1995,1997), Wang (1999) and Ke (2005). Many researchers have studied the queueing model with unreliable server in different frameworks and suggested ways and means to tackle related situations. Grey (2000) incorporated the server breakdown on vacation queueing model. Haridass and Arumuganathan (2008) studied $M^{[X]}/G/1$ queueing system with an unreliable server and with single vacation. Choudhury and Deka(2012) investigated an $M/G/1$ unreliable server Bernoulli vacation queue with two phases of service. In 2013, the same authors studied a batch arrival unreliable server Bernoulli vacation queue with two phases of service and delayed repair. Ke (2012), analyzed an $M^{[X]}/G/1$ queueing system with an unreliable server and repair, in which the server operates with a randomized vacation policy with multiple available vacation. Kalyanaraman and Nagarajan (2016), analysed a batch arrival, fixed batch service queue with unreliable server and with Bernoulli server vacation. Doshi (1991), analysed a queueing system in which each customers receives two services. Madan (2001) considered a single server queue with two stages of heterogeneous service.

In this article we consider an $M^{[X]}/G^K/1$ queue with unreliable server and with Bernoulli vacation. In addition each service has two stages. This type of queueing system exists in manufacturing industries, Transportation system etc. In manufacturing industries, after products are approved for transportation to customer shops, they are transported to the shops in bulks by truck. After transporting the products, if no batch is available for transportation, the truck will be used for other work or the truck is sent for maintenance (vacation period). During the service period (transportation period), the trucks may break down. The above situation can be modeled as an $M^{[X]}/G^K/1$ queue with unreliable server and Bernoulli vacation.

The remainder of this article is organized as follows: Section 2 provides the model description and mathematical analysis. In section 3, we obtain some queueing characteristics of the model discussed in this paper. In section 4, we present some particular models. In section 5, we illustrate the model by some numerical examples. Finally, In section 6 we present a conclusion.

The Model And Analysis: We consider and $M^{[X]}/G^K/1$ queueing system, where the number of customers arrives to the system at time instant follows a compound Poisson process with arrival rate λ . The size of the successive arriving batches is a random variable with probability $P\{X=j\}=C_j$, whose probability generating function is defined

$$\text{by } C(z) = \sum_{j=1}^{\infty} C_j z^j.$$

The services are given in batches of fixed size 'K'. Each customer undergoes two stages of heterogeneous service provided by a single server on a first come first served basis. The service times of the two stages follow different generally distributed random variables with distribution function $G_i(x)$ and the density function $g_i(x)$ for $i=0,1$. After completion of second stage of service, the server takes a Bernoulli vacation of random duration. The vacation period is also generally distributed with distribution function $B(x)$. In addition, the server may breakdown

during a service and the breakdowns are assumed to occur according to a Poisson process with the rate ' α '. Once the server breakdown, the customer whose service is interrupted goes to the head of the queue and the repair to server starts immediately. The duration of the repair period is generally distributed with distribution function $H(x)$. Immediately after the broken server is repaired, the server is ready to start its service. Further, we assume that the input process, server life time, server repair time, service time and vacation times are independent of each other.

The analysis of this model is based on supplementary variable technique and the supplementary variable is elapsed service time / elapsed vacation time / elapsed repair time.

We define the following probabilities and conditional probabilities:

$\mu_i(x) = \frac{g_i(x)}{1 - G_i(x)}$ for $i=1,2$ is the conditional probability that the completion of i^{th} phase service during the

interval $(x, x + dx)$, given that the elapsed service time is ' x '.

$\beta(x) = \frac{b(x)}{1 - B(x)}$ is the conditional probability that the completion of vacation during the interval $(x, x + dx)$,

given that the elapsed vacation time is ' x '.

$\gamma(x) = \frac{h(x)}{1 - H(x)}$ is the conditional probability that the completion of repair during the interval $(x, x + dx)$,

given that the elapsed repair time is ' x '.

The Markov process related to this model is $\{(N(t), S(t)) : t \geq 0\}$ where $N(t)$ be the number of customer in the queue and $S(t)$ be the supplementary variable at time t . and

$S(t) = S_{1_i}(t)$, elapsed i^{th} phase service time $i=1,2$.

= $S_2(t)$, the elapsed vacation time

= $S_3(t)$, the elapsed repair time

$P_n^{(i)}(t, x)$ = Probability that, at time ' t ', there are ' n ' customers in the queue, the server provides the ' i ' stage of service and (excluding the customer in service) the elapsed service time is ' x '. where $i=1,2$.

$V_n(t, x)$ = Probability that, at time ' t ', there are ' n ' customers in the queue and the elapsed vacation time is ' x '

$R_n(t, x)$ = Probability that, at time ' t ', there are ' n ' customers in the queue and the elapsed repair time is ' x ' and

$Q_n(t)$ = Probability that, at time ' t ', there are n customers in the queue and the server is idle

The differential-difference equations for this model are

$$\frac{dP_0^{(1)}(x)}{dx} = -(\lambda + \mu_1(x) + \alpha)P_0^{(1)}(x) \quad (1)$$

$$\frac{dP_n^{(1)}(x)}{dx} = -(\lambda + \mu_1(x) + \alpha)P_n^{(1)}(x) + \lambda \sum_{j=1}^n C_j P_{n-j}^{(1)}(x), \quad \text{for } n=0,1,\dots \quad (2)$$

$$\frac{dP_0^{(2)}(x)}{dx} = -(\lambda + \mu_2(x) + \alpha)P_0^{(2)}(x) \quad (3)$$

$$\frac{dP_n^{(2)}(x)}{dx} = -(\lambda + \mu_2(x) + \alpha)P_n^{(2)}(x) + \lambda \sum_{j=1}^n C_j P_{n-j}^{(2)}(x), \quad \text{for } n=0,1,\dots \quad (4)$$

$$\frac{dV_0(x)}{dx} = -(\lambda + \beta(x))V_0(x) \quad (5)$$

$$\frac{dV_n(x)}{dx} = -(\lambda + \beta(x))V_n(x) + \lambda \sum_{j=1}^n C_j V_{n-j}(x), \quad \text{for } n=0,1,\dots \quad (6)$$

$$\frac{dR_0(x)}{dx} = -(\lambda + \gamma(x))R_0(x) \quad (7)$$

$$\frac{dR_n(x)}{dx} = -(\lambda + \gamma(x))R_n(x) + \lambda \sum_{j=1}^n C_j R_{n-j}(x), \quad \text{for } n=1, \dots \quad (8)$$

$$0 = -\lambda Q_n + \lambda(1 - \delta_{n,K}) \sum_{j=1}^n c_j Q_{n-j} + \int_0^\infty R_n(x) \gamma(x) dx + \int_0^\infty V_n(x) \beta(x) dx + (1-p) \int_0^\infty P_n^{(2)}(x) \mu_2(x) dx \quad (9)$$

The boundary conditions are

$$P_n^{(1)}(0) = \int_0^\infty V_{n+K}(x) \beta(x) dx + \int_0^\infty R_{n+K}(x) \gamma(x) dx + (1-p) \int_0^\infty P_{n+K}^{(2)}(x) \mu_2(x) dx + \lambda \sum_{j=0}^{K-1} C_{n+K-j} Q_j, \quad \text{for } n=0, 1, \dots \quad (10)$$

$$P_n^{(2)}(0) = \int_0^\infty P_n^{(1)}(x) \mu_1(x) dx, \quad \text{for } n=0, 1, \dots \quad (11)$$

$$V_n(0) = p \int_0^\infty P_n^{(2)}(x) \mu_2(x) dx, \quad \text{for } n=0, 1, \dots \quad (12)$$

$$R_n(0) = \alpha \int_0^\infty (P_{n-K}^{(1)}(x) + P_{n-K}^{(2)}(x)) dx, \quad \text{for } n=K, K+1, \dots \quad (13)$$

$$R_n(0) = 0, \quad \text{for } n=0, 1, 2, \dots, K-1 \quad (14)$$

and the normalization condition is

$$\sum_{n=0}^{K-1} Q_n + \int_0^\infty \sum_{n=0}^\infty [P_n^{(1)}(x) + P_n^{(2)}(x) + V_n(x) + R_n(x)] dx = 1 \quad (15)$$

For the analysis, We define the following probability generating functions

$$P_1(x, z) = \sum_{n=0}^\infty P_n^{(1)}(x) z^n, \quad P_2(x, z) = \sum_{n=0}^\infty P_n^{(2)}(x) z^n, \quad R(x, z) = \sum_{n=0}^\infty R_n(x) z^n, \quad C(z) = \sum_{j=0}^\infty C_j z^j, \quad ,$$

$$Q(z) = \sum_{n=0}^{K-1} Q_n z^n, \quad V(x, z) = \sum_{n=0}^\infty V_n(x) z^n.$$

Multiplying equation (2) by z^n and applying $\sum_{n=1}^\infty$, we have

$$\frac{\partial \sum_{n=1}^\infty P_n^{(1)}(x) z^n}{\partial x} = -(\lambda + \mu_1(x) + \alpha) \sum_{n=1}^\infty P_n^{(1)}(x) z^n + \lambda \sum_{n=1}^\infty \sum_{j=1}^n C_j P_{n-j}^{(1)}(x) z^n$$

Adding the above equation with equation (1), we have

$$\frac{\partial P_1(x, z)}{\partial x} + (\lambda - \lambda C(z) + \mu_1(x) + \alpha) P_1(x, z) = 0 \quad (16)$$

Multiplying equation (4) by z^n and applying $\sum_{n=1}^\infty$, we have

$$\frac{\partial \sum_{n=1}^\infty P_n^{(2)}(x) z^n}{\partial x} = -(\lambda + \mu_2(x) + \alpha) \sum_{n=1}^\infty P_n^{(2)}(x) z^n + \lambda \sum_{n=1}^\infty \sum_{j=1}^n C_j P_{n-j}^{(2)}(x) z^n$$

Adding the above equation with equation (3), we have

$$\frac{\partial P_2(x, z)}{\partial x} + (\lambda - \lambda C(z) + \mu_2(x) + \alpha) P_2(x, z) = 0 \quad (17)$$

Multiplying equation (6) by z^n and applying $\sum_{n=1}^\infty$, we have

$$\frac{\partial \sum_{n=1}^\infty V_n(x) z^n}{\partial x} = -(\lambda + \beta(x)) \sum_{n=1}^\infty V_n(x) z^n + \lambda \sum_{n=1}^\infty \sum_{j=1}^n C_j V_{n-j}(x) z^n$$

Adding the above equation with equation (5), we have

$$\frac{\partial V(x, z)}{\partial x} + (\lambda - \lambda C(z) + \beta(x))V(x, z) = 0 \quad (18)$$

Multiplying equation (8) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial \sum_{n=1}^{\infty} R_n(x) z^n}{\partial x} = -(\lambda + \gamma(x)) \sum_{n=1}^{\infty} R_n(x) z^n + \lambda \sum_{n=1}^{\infty} \sum_{j=1}^n C_j R_{n-j}(x) z^n$$

Adding the above equation with equation(7), we have

$$\frac{\partial R(x, z)}{\partial x} + (\lambda - \lambda C(z) + \gamma(x))R(x, z) = 0 \quad (19)$$

Multiplying equation (10) by z^{n+K} and applying $\sum_{n=0}^{\infty}$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} P_n^{(1)}(0) z^{n+K} &= \lambda \sum_{n=0}^{\infty} \sum_{j=0}^{K-1} Q_j C_{n+K-j} z^{n+K} + \int_0^{\infty} \gamma(x) \sum_{n=0}^{\infty} R_{n+K}(x) z^{n+K} dx + \int_0^{\infty} \beta(x) \sum_{n=0}^{\infty} V_{n+K}(x) z^{n+K} dx \\ &+ (1-p) \int_0^{\infty} \mu_2(x) \sum_{n=0}^{\infty} P_{n+K}^{(2)}(x) z^{n+K} dx \\ \Rightarrow z^K P_1(0, z) &= \int_0^{\infty} \gamma(x) \sum_{n=K}^{\infty} R_n(x) z^n dx + (1-p) \int_0^{\infty} \mu_2(x) \sum_{n=K}^{\infty} P_n^{(2)}(x) z^n dx \\ &+ K(z) + \int_0^{\infty} \beta(x) \sum_{n=K}^{\infty} V_n(x) z^n dx \quad (20) \end{aligned}$$

where $K(z) = \lambda \sum_{n=0}^{\infty} \sum_{j=0}^{K-1} Q_j C_{n+K-j} z^{n+K}$

We multiplying equation (9) by z^n and applying $\sum_{n=0}^{K-1}$, we get

$$\begin{aligned} 0 &= -\lambda \sum_{n=0}^{K-1} Q_n z^n + \int_0^{\infty} \gamma(x) \sum_{n=0}^{K-1} R_n(x) z^n dx + \int_0^{\infty} \beta(x) \sum_{n=0}^{K-1} V_n(x) z^n dx + \lambda \sum_{n=0}^{K-1} (1 - \delta_{n,K}) \sum_{j=1}^n C_j Q_{n-j} z^n \\ &+ (1-p) \int_0^{\infty} \mu_2(x) \sum_{n=0}^{K-1} P_n^{(2)}(x) z^n dx \\ 0 &= -\lambda Q(z) + \int_0^{\infty} \gamma(x) \sum_{n=0}^{K-1} R_n(x) z^n dx + \int_0^{\infty} \beta(x) \sum_{n=0}^{K-1} V_n(x) z^n dx + \lambda L(z) \\ &+ (1-p) \int_0^{\infty} \mu_2(x) \sum_{n=0}^{K-1} P_n^{(2)}(x) z^n dx \quad (21) \end{aligned}$$

where $L(z) = \sum_{n=0}^{K-1} (1 - \delta_{n,K}) \sum_{j=1}^n C_j Q_{n-j} z^n$

We add the equations (20) and (21), we get

$$\begin{aligned} z^K P_1(0, z) &= \int_0^{\infty} \beta(x) V(x, z) dx + \int_0^{\infty} \gamma(x) R(x, z) dx - \lambda Q(z) + K(z) + \lambda L(z) \\ &+ (1-p) \int_0^{\infty} \mu_2(x) P_2(x, z) dx \\ \text{where } K(z) &= \lambda [C(z)Q(z) - L(z)] \\ z^K P_1(0, z) &= \int_0^{\infty} \beta(x) V(x, z) dx + \int_0^{\infty} \gamma(x) R(x, z) dx - \lambda Q(z) + [\lambda C(z)Q(z) - \lambda L(z)] + \lambda L(z) \\ &+ (1-p) \int_0^{\infty} \mu_2(x) P_2(x, z) dx \\ z^K P_1(0, z) &= \int_0^{\infty} \beta(x) V(x, z) dx + \int_0^{\infty} \gamma(x) R(x, z) dx + (1-p) \int_0^{\infty} \mu_2(x) P_2(x, z) dx \\ &+ \lambda [C(z) - 1] Q(z) \quad (22) \end{aligned}$$

Multiplying equation (11) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$\sum_{n=0}^{\infty} P_n^{(2)}(0)z^n = \int_0^{\infty} \mu_1(x) \sum_{n=0}^{\infty} P_n^{(1)}(x)z^n dx$$

$$P_2(0, z) = \int_0^{\infty} \mu_1(x) P_1(x, z) dx \quad (23)$$

Multiplying equation (12) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$V(0, z) = p \int_0^{\infty} \mu_2(x) P_2(x, z) dx \quad (24)$$

Multiplying equation (13) by z^n and applying $\sum_{n=K}^{\infty}$, we have

$$\sum_{n=K}^{\infty} R_n(0)z^n = \alpha \int_0^{\infty} \left(\sum_{n=K}^{\infty} P_{n-K}^{(1)}(x)z^n + \sum_{n=K}^{\infty} P_{n-K}^{(2)}(x)z^n \right) dx$$

Adding the above equation with equation (14), we have

$$R(0, z) = \alpha z^K \int_0^{\infty} (P_1(x, z) + P_2(x, z)) dx = \alpha z^K (P_1(z) + P_2(z)) \quad (25)$$

Integrating equation (16) partially with respect to 'x' with the limits from '0' to 'x', we have

$$P_1(x, z) = P_1(0, z) e^{-ax - \int_0^x \mu_1(x) dx} \quad (26)$$

where $a = \lambda - \lambda C(z) + \alpha$

Integrating equation (26) partially with respect to 'x' with the limits from '0' to ' ∞ ', we have

$$P_1(z) = \frac{P_1(0, z)[1 - G_1^*(a)]}{a}, \quad (27)$$

Multiplying equation (26) by $\mu(x)$ and integrating partially with respect to 'x', with the limits from '0' to ' ∞ ', we have

$$\int_0^{\infty} \mu_1(x) P_1(x, z) dx = P_1(0, z) G_1^*(a), \quad (28)$$

Integrating equation (17) partially with respect to 'x' with the limits from '0' to 'x', we have

$$P_2(x, z) = P_2(0, z) e^{-ax - \int_0^x \mu_2(x) dx} \quad (29)$$

where $a = \lambda - \lambda C(z) + \alpha$

Integrating equation (29) partially with respect to 'x' with the limits from '0' to ' ∞ ', we have

$$P_2(z) = \frac{P_2(0, z)[1 - G_2^*(a)]}{a}, \quad (30)$$

Multiplying equation (29) by $\mu(x)$ and integrating partially with respect to 'x', with the limits from '0' to ' ∞ ', we have

$$\int_0^{\infty} \mu_2(x) P_2(x, z) dx = P_2(0, z) G_2^*(a), \quad (31)$$

Substituting equation (31) in (24), we have $V(0, z) = p P_2(0, z) G_2^*(a)$ (32)

Integrating equation (18) partially with respect to 'x', with the limits from '0' to 'x', we have

$$V(x, z) = V(0, z) e^{-mx - \int_0^x \beta(x) dx} \quad (33)$$

where $m = \lambda - \lambda C(z)$

Substituting equation (32) in equation (33), we get

$$V(x, z) = p P_2(0, z) G_2^*(a) e^{-mx - \int_0^x \beta(x) dx} \quad (34)$$

Integrating equation (34) partially with respect to 'x', with the limits from '0' to ' ∞ ', we have

$$V(z) = \frac{pP_2(0,z)G_2^*(a)[1-B^*(m)]}{m} \quad (35)$$

Multiplying equation (34) by $\beta(x)$ and integrating partially with respect to x , with limits from 0 to ∞ .

$$\int_0^\infty V(x,z)\beta(x)dx = pP_2(0,z)G_2^*(a)B^*(m) \quad (36)$$

Integrating equation (19) partially with respect to ' x ', with the limits from '0' to ' x ', we have

$$R(x,z) = R(0,z)e^{-mx - \int_0^x \gamma(x)dx} \quad (37)$$

Substituting equation (25),(27) and (30) in (29), we have

$$R(x,z) = \frac{\alpha z^K e^{-mx - \int_0^x \gamma(x)dx}}{a} [P_1(0,z)[1-G_1^*(a)] + P_2(0,z)[1-G_2^*(a)]] \quad (38)$$

Integrating equation (38) partially with respect to ' x ', with the limits from 0 to ∞ , we have

$$R(z) = \frac{\alpha z^K [1-H^*(m)]}{am} [P_1(0,z)(1-G_1^*(a)) + P_2(0,z)(1-G_2^*(a))] \quad (39)$$

Multiplying equation (38) by $\gamma(x)$ and integrating partially with respect to ' x ', with the limits from 0 to ∞ , we have

$$\int_0^\infty R(x,z)\gamma(x)dx = \frac{\alpha z^K H^*(m)}{a} [P_1(0,z)(1-G_1^*(a)) + P_1(0,z)G_1^*(a)(1-G_2^*(a))] \quad (40)$$

Substitute equation (28) in (23), we have

$$P_2(0,z) = P_1(0,z)G_1^*(a) \quad (41)$$

Now using equation (40), (41), (36) and (31) in equation (22), we have

$$P_1(0,z) = \frac{aQ(z)m}{D} \quad (42)$$

where $D = \alpha z^K [1-G_1^*(a)G_2^*(a)]H^*(m) - a[z^K - (1-p + pB^*(m))G_1^*(a)G_2^*(a)]$

Substituting $P_1(0,z)$ in the equation (41), we have

$$P_2(0,z) = \frac{aQ(z)mG_1^*(a)}{D} \quad (43)$$

Substituting $P_1(0,z)$ in the equation (27), we have

$$P_1(z) = \frac{mQ(z)[1-G_1^*(a)]}{D} \quad (44)$$

Substituting $P_2(0,z)$ in the equation (30), we have

$$P_2(z) = \frac{mQ(z)[1-G_2^*(a)]G_1^*(a)}{D} \quad (45)$$

Substituting $P_2(0,z)$ in the equation (35), we have

$$V(z) = \frac{paG_1^*(a)G_2^*(a)[1-B^*(m)]Q(z)}{D} \quad (46)$$

Substituting $P_1(0,z)$ & $P_2(0,z)$ in the equation (39), we have

$$R(z) = \frac{\alpha z^K Q(z)[1-G_1^*(a)G_2^*(a)][1-H^*(m)]}{D} \quad (47)$$

Now adding (44), (45), (46) and (47), we have

$$S(z) = P_1(z) + P_2(z) + V(z) + R(z) \quad (48)$$

Here $S(z)$ represent the probability generating function of number of customer in the queue, independent of server state

$$S(z) = \frac{N}{D} \quad (49)$$

$$N = Q(z)\{apG_1^*(a)G_2^*(a)[1-B^*(m)] + [m + \alpha z^K(1-H^*(m))][1-G_1^*(a)G_2^*(a)]\}$$

where $m = \lambda - \lambda C(z)$, $a = \lambda - \lambda C(z) + \alpha$

We know that $S(z)$ is probability generating function, it has the property that it must converge inside the unit circle $|z| = 1$. Here it can be seen that the expression in the denominator of $S(z)$ has 'K' zero. By Rouches theorem, we notice that K-1 zero's of this expression lies inside the unit circle $|z| = 1$, and must coincide with K-1 zero's of numerator of $S(z)$, and one zero lies out side the unit circle $|z| = 1$. Let z_0 be the zero which lies outside the circle $|z| = 1$. As $S(z)$ converges, K-1 zero's of numerator and denominator of $S(z)$ will be cancelled. Therefore

$$\text{we have } S(z) = \frac{A}{z - z_0} \text{ By substituting } z=1, \text{ we get } A = (1 - z_0)S(1)$$

$$S(1) = \frac{N_1}{D_1} \quad (39)$$

$$N_1 = Q\lambda E(X)\{[1-G_1^*(\alpha)G_2^*(\alpha)][1+\alpha E(R)] + \alpha pG_1^*(\alpha)G_2^*(\alpha)E(V)\}$$

$$D_1 = \alpha G_1^*(\alpha)G_2^*(\alpha)[K - \lambda pE(X)E(V)] - \lambda E(X)[1-G_1^*(\alpha)G_2^*(\alpha)][1+\alpha E(R)]$$

$$\text{Substituting the value of } S(1) \text{ in the above equation, we get } A = \frac{(1 - z_0)N_1}{D_1} \quad (40)$$

$$\text{Substituting the value of } A \text{ in equation (38), we get } S(z) = \frac{(z_0 - 1)N_1}{z_0 D_1} \sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n \quad (41)$$

Which is probability generating function of number of customer in the queue.

System Performance Measures: In this section, the system performance measures, the mean number of customers in the queue and idle probability have been presented.

(i) The mean number of customers in the queue

$$E(N) = S'(1) = \frac{z_0 N_1}{(z_0 - 1)D_1} \quad (42)$$

(ii) The idle probability

Since $Q + S(1) = 1$, where $Q = \sum_{n=0}^{K-1} Q_n$, which lead to

$$Q = 1 - \lambda E(X) \left\{ \frac{1}{\alpha K G_1^*(\alpha) G_2^*(\alpha)} - \frac{1}{\alpha K} + \frac{E(R)}{K G_1^*(\alpha) G_2^*(\alpha)} - \frac{E(R)}{K} + \frac{pE(V)}{K} \right\} \quad (43)$$

Somr Particular Models: In this section, four particular models have been obtained by assigning particular forms to the parameters and to the distribution function.

Particular Model-01: In the above model, we assume that batch arrival size random variable X follows geometric distributin with probablilty $C_n = (1-s)^{n-1}s$ for $n \geq 1$ and $s=1-t$, then $E(X) = \frac{1}{s}$. Also we assume that the service

time random variables follows exponential distribution with $E(S) = \frac{1}{\mu_i}$ then $G^*(\alpha) = \frac{\mu_i}{\alpha + \mu_i}$, for $i=1,2$ and the

repair time random variable R follows exponential distribution with $E(R) = \frac{1}{\gamma}$, In addition, we assume that vacation

time random variable V follows exponential distribution with $E(V) = \frac{1}{\beta}$, Now equations (41), (42) and (43) becomes

The probability generating function of number of customers in the queue

$$S(z) = \sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n \frac{(z_0 - 1)Q\lambda[\beta(\alpha + \gamma)(\alpha + \mu_1 + \mu_2) + p\mu_1\mu_2\gamma]}{z_0\{\mu_1\mu_2\gamma(Ks\beta - p\lambda) - \lambda\beta(\alpha + \gamma)(\alpha + \mu_1 + \mu_2)\}}$$

The idle probability

$$Q = \frac{\left\{ \begin{array}{l} \beta\mu_1\mu_2\gamma Ks \\ -\lambda[\beta(\alpha + \gamma)(\alpha + \mu_1 + \mu_2) + p\mu_1\mu_2\gamma] \end{array} \right\}}{\beta\mu_1\mu_2\gamma Ks}$$

The mean number of customers in the queue

$$E(N) = \frac{z_0 Q \lambda [(\alpha + \gamma)\beta(\alpha + \mu_1 + \mu_2) + p\mu_1\mu_2\gamma]}{\left\{ \begin{array}{l} (z_0 - 1) \{ \mu_1\mu_2\gamma [Ks\beta - p\lambda] \} \\ -\beta\lambda(\alpha + \gamma)(\alpha + \mu_1 + \mu_2) \} \right\}}$$

Particular Model -02: If we put $p=0$, we get the model without vacation.

The probability generating function of number of customers in the queue

$$S(z) = \frac{Q(z) \{ [m + \alpha z^K (1 - H^*(m))] [1 - G_1^*(a)G_2^*(a)] \}}{\left\{ \begin{array}{l} \alpha z^K [1 - G_1^*(a)G_2^*(a)] H^*(m) \\ -a[z^K - G_1^*(a)G_2^*(a)] \end{array} \right\}}$$

where $m = \lambda - \lambda C(z)$, $a = \lambda - \lambda C(z) + \alpha$

The idle probability

$$Q = 1 - \lambda E(X) \left\{ \frac{1}{\alpha K G_1^*(\alpha) G_2^*(\alpha)} - \frac{1}{\alpha K} + \frac{E(R)}{K G_1^*(\alpha) G_2^*(\alpha)} - \frac{E(R)}{K} \right\}$$

The mean number of customers in the queue

$$E(N) = \frac{(z_0) Q \lambda E(X) \{ [1 - G_1^*(\alpha)G_2^*(\alpha)] [1 + \alpha E(R)] \}}{(z_0 - 1) \left\{ \begin{array}{l} \alpha G_1^*(\alpha) G_2^*(\alpha) K \\ -\lambda E(X) [1 - G_1^*(\alpha)G_2^*(\alpha)] [1 + \alpha E(R)] \end{array} \right\}}$$

Particular Model -03: If we put $K=1$, we get a model with batch service of size one.

The probability generating function of number of customers in the queue,

$$S(z) = \frac{QJ_1}{J_2}$$

$$J_1 = \{ [m + \alpha z(1 - H^*(m))] [1 - G_1^*(a)G_2^*(a)] + apG_1^*(a)G_2^*(a)[1 - B^*(m)] \}$$

$$J_2 = \alpha z [1 - G_1^*(a)G_2^*(a)] H^*(m) - a[z - (1 - p + pB^*(m))G_1^*(a)G_2^*(a)]$$

where $m = \lambda - \lambda C(z)$, $a = \lambda - \lambda C(z) + \alpha$

The Idle probability

$$Q = 1 - \lambda E(X) \left\{ \frac{1}{\alpha G_1^*(\alpha) G_2^*(\alpha)} - \frac{1}{\alpha} - E(R) + \frac{E(R)}{G_1^*(\alpha) G_2^*(\alpha)} + pE(V) \right\}$$

The mean number of customers in the queue

$$E(N) = \frac{z_0 Q \lambda E(X) L_1}{(z_0 - 1) L_2}$$

$$L_1 = \{ [1 - G_1^*(\alpha)G_2^*(\alpha)] [1 + \alpha E(R)] + apG_1^*(\alpha)G_2^*(\alpha)E(V) \}$$

$$L_2 = \{ \alpha G_1^*(\alpha)G_2^*(\alpha) [1 - \lambda p E(X) E(V)] - \lambda E(X) [1 - G_1^*(\alpha)G_2^*(\alpha)] [1 + \alpha E(R)] \}$$

PARTICULAR MODEL -04: If we put $k=1$, and $X=1$, we get a model with single arrival and batch size of one.

The probability generating function of number of customers in the queue

$$S(z) = \frac{QJ_1}{J_2}$$

$$J_1 = \{ [m + \alpha z(1 - H^*(m))] [1 - G_1^*(a)G_2^*(a)] + apG_1^*(a)G_2^*(a)[1 - B^*(m)] \}$$

$$J_2 = \alpha z [1 - G_1^*(a)G_2^*(a)] H^*(m) - a[z - (1 - p + pB^*(m))G_1^*(a)G_2^*(a)]$$

where $m = \lambda - \lambda z$, $a = \lambda - \lambda z + \alpha$

The idle probability

$$Q = 1 - \lambda \left\{ \frac{1}{\alpha G_1^*(\alpha) G_2^*(\alpha)} - \frac{1}{\alpha} + \frac{E(R)}{G_1^*(\alpha) G_2^*(\alpha)} - E(R) + E(V) \right\}$$

The mean number of customers in the queue

$$E(N) = \frac{z_0 Q \lambda L_1}{(z_0 - 1) L_2}$$

$$L_1 = \{ [1 - G_1^*(\alpha) G_2^*(\alpha)] [1 + \alpha E(R)] + \alpha p G_1^*(\alpha) G_2^*(\alpha) E(V) \}$$

$$L_2 = \{ \alpha G_1^*(\alpha) G_2^*(\alpha) [1 - \lambda p E(V)] - \lambda [1 - G_1^*(\alpha) G_2^*(\alpha)] [1 + \alpha E(R)] \}$$

Numerical Example: In this section, we present some numerical examples related to the models in section 4. We fix the values of β , α , γ , μ_1 , μ_2 , K , s , p and we vary the values of the arrival rate λ . For various values of z_0 , we find the values of $E(N)$. Also we find the values of Q . The results are presented in tables 1 to 4, respectively for models 01 to 04. From the values, it is clear that, as the arrival rate increases, the idle probability decreases. Which is very much coincide with our expectations. Also the mean number of customers in the queue increases, for increasing values of arrival rate. Again, which is very much coincide with our expectation. Surprisingly in all the models, If the zero z_0 increases from 1.00001 to 15, the mean number of customers in the queue considerably decreases.

Table.1. Q and E(N) for model 01 ($\beta = 2, \alpha = 1, \gamma = 1, \mu_1 = 6, \mu_2 = 6, K = 15, s = 0.7, p = 0.5$)

λ	Q	E(N)					
		Z_0 values					
		1.000.	1.00001	1.5	5	10	15
1	0.9074	926	9259.4	0.2778	0.1157	0.1029	0.0992
2	0.8148	1852	18519	0.5556	0.2315	0.2058	0.1984
3	0.7222	2778.1	27777	0.8333	0.3472	0.3086	0.2976
4	0.6296	3704.1	37037	1.1111	0.4630	0.4115	0.3968
5	0.5370	4630.1	46297	1.3889	0.5787	0.5144	0.4960
6	0.4444	5556.1	55556	1.6667	0.6944	0.6173	0.5952
7	0.3519	6482.1	64815	1.9444	0.8102	0.7202	0.6944
8	0.2593	7408.1	74075	2.2222	0.9259	0.8230	0.7937
9	0.1667	8334.2	83334	2.5000	1.0417	0.9259	0.8929
10	0.0741	9260.2	92594	2.7778	1.1574	1.0288	0.9921

Table.2. Q and E(N) for model 02 ($\beta = 20, \alpha = 10, \gamma = 10, \mu_1 = 60, \mu_2 = 60, K = 10, s = 0.9$)

λ	Q	E(N)					
		Z_0 values					
		1.000.	1.00001	1.5	5	10	15
1	0.9920	80.2	802.4	0.0241	0.0100	0.0089	0.0086
2	0.9840	160.5	1605.0	0.0481	0.0201	0.0178	0.0172
3	0.9759	240.7	2407.4	0.0722	0.0301	0.0267	0.0258
4	0.9679	321.0	3209.9	0.0963	0.0401	0.0357	0.0344
5	0.9599	401.2	4012.4	0.1204	0.0502	0.0446	0.0430
6	0.9519	481.5	4814.9	0.1444	0.0602	0.0535	0.0516
7	0.9438	561.7	5617.3	0.1685	0.0702	0.0624	0.0602
8	0.9358	642.0	6419.8	0.1926	0.0802	0.0713	0.0688
9	0.9278	722.2	7222.3	0.2167	0.0903	0.0802	0.0774
10	0.9198	802.5	8024.8	0.2407	0.1003	0.0892	0.0860

Table.3. Q and E(N) for model 03 ($\beta = 20, p = 0.5, \alpha = 10, \gamma = 10, \mu_1 = 60, \mu_2 = 60, K = 10, s = 0.7$)

λ	Q	E(N)					
		Z_0 values					
		1.000.	1.00001	1.5	5	10	15

1	0.9861	138.9	1388.9	0.0364	0.0152	0.0154	0.0149
2	0.9722	277.8	2777.8	0.0738	0.0308	0.0309	0.0298
3	0.9583	416.7	4166.7	0.1123	0.0468	0.0463	0.0446
4	0.9444	555.6	5555.6	0.1520	0.0633	0.0617	0.0595
5	0.9306	694.5	6944.5	0.1928	0.0803	0.0772	0.0744
6	0.9167	833.4	8333.4	0.2348	0.0979	0.0926	0.0893
7	0.9028	972.3	9722.3	0.2782	0.1159	0.1080	0.1042
8	0.8889	1111.2	11111	0.3229	0.1345	0.1235	0.1190
9	0.8750	1250.1	12500	0.3690	0.1538	0.1389	0.1339
10	0.8611	1389.0	13669	0.4167	0.1736	0.1543	0.1488

Table.4. Q and E(N) for model 4 ($\beta = 20, p = 0.5, \alpha = 10, \gamma = 10, \mu_1 = 60, \mu_2 = 60, K = 10, s = 0.7$)

λ	Q	E(N)					
		Z_0 values					
		1.000.	1.00001	1.5	5	10	15
1	0.9903	97.2	972.2	0.0292	0.0122	0.0108	0.0104
2	0.9806	194.4	1944.5	0.0583	0.0243	0.0216	0.0208
3	0.9708	291.6	2916.7	0.0875	0.0365	0.0324	0.0313
4	0.9611	388.9	3888.9	0.1167	0.0486	0.0432	0.0417
5	0.9514	486.1	4861.2	0.1458	0.0608	0.0540	0.0521
6	0.9417	583.3	5833.4	0.1750	0.0729	0.0648	0.0625
7	0.9319	680.6	6805.6	0.2042	0.0851	0.0756	0.0729
8	0.9222	777.8	7777.9	0.2333	0.0972	0.0864	0.0833
9	0.9125	875.0	8750.1	0.2625	0.1094	0.0972	0.0938
10	0.9028	972.3	9722.3	0.2917	0.1215	0.1080	0.1042

CONCLUSION

In this article, a single server batch arrival, batch service (fixed) queue with Bernoulli vacation and with unreliable server has been completely analysed. In addition the server provides two stages of service. To illustrate the analytical compatibility of the model we present some numerical examples by taking particular values to the parameters and particular form to the probability distribution. The model can be extended by taking the break down period as generally distributed.

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